Lecture notes Models of Mechanics

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Lecture 7: Small deformation theories
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Small deformation theories

Well known theories of strength-of-materials (Hållfasthetslära) are based on a small displacement assumption. There is essentially two ways of deriving such theories:

**Rigorous** Derive the (non-linear) large deformation theory and perform a (mathematical) linearization.

**Direct** Identify (in one or several simple examples) consequences of the linearization and use these as assumptions.
The assumptions are:

- Equations of motion (equilibrium) are stated in a reference configuration that has zero displacements (e.g., we ignore the deformation when formulating equilibrium).
- Constitutive laws (specific laws) are stated assuming that strain is a linear function of displacement, and stress is a linear function of strain.
All small displacement linear theories have a common structure that can be described by three equations:

- \( e = Du \) “strain is a linear function of the displacement”
- \( f = D^T \sigma \) equilibrium equation “force and stress are linearly related by a constant operator”
- \( \sigma = Ee \) “stress and strain are linearly related”

Stress and strain are defined such that it is “the same” operator (transposed when matrices) that appears in both of the first two equations. This is related to the existence of a “work” identity:

\[
f^T u = (D^T \sigma)^T u = \sigma^T Du = \sigma^T e, \quad \text{external work} = \text{internal work}
\]
Combining the three equations we get

\[ f = D^T E D \vec{u} \]

when the operators are matrices

\[ K = D^T E D \]

is known as the **stiffness matrix**. It is symmetric and (usually) positive definite.
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Equilibrium:

\[ f = D^T \sigma \quad \iff \]

\[ \frac{\partial f_n}{\partial s} + q_n = 0, \quad (1) \]

\[ \frac{\partial f_t}{\partial s} + q_t = 0, \quad (2) \]

\[ f_n + \frac{\partial m_b}{\partial s} + l_b = 0. \quad (3) \]
Geometry:

\[ \sigma = E e \quad \iff \]

\[ \varepsilon_n = \frac{\partial u_n}{\partial s} - \varphi \quad \text{(shearing)}, \quad (4) \]

\[ \varepsilon_t = \frac{\partial u_t}{\partial s} \quad \text{(elongation)}, \quad (5) \]

\[ \omega = \frac{\partial \varphi}{\partial s} \quad \text{(bending)}. \quad (6) \]
Constitutive law:

\[ \sigma = E e \quad \iff \quad f_n = k \varepsilon_n, \]
\[ f_t = (EA) \varepsilon_t, \]  \hspace{1cm} (7)
\[ m_b = (EI) \omega, \] \hspace{1cm} (9)
Small deformation theories – beam theory

By eliminating $f_n$, $f_t$, $m_b$, $\varepsilon_n$, $\varepsilon_t$ and $\omega$ we obtain the following equations, which provide the direct coupling between $u_n$, $u_t$ and $\varphi$, on one hand, and $q_n$, $q_t$ and $l_b$, on the other:

\[
\frac{\partial}{\partial s} \left( (EA) \frac{\partial u_t}{\partial s} \right) + q_t = 0,
\]
\[
\frac{\partial}{\partial s} \left( k \left( \frac{\partial u_n}{\partial s} - \varphi \right) \right) + q_n = 0,
\]
\[
k \left( \frac{\partial u_n}{\partial s} - \varphi \right) + \frac{\partial}{\partial s} \left( (EI) \frac{\partial \varphi}{\partial s} \right) + l_b = 0.
\]
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Bar problem: Given a distribution of force per unit length $q_t = q_t(s)$, $s \in (0, L)$, and at each end of the beam natural or essential boundary conditions, find the displacement field $u_t = u_t(s)$ such that (10) is satisfied for all $s \in (0, L)$.

Timoshenko beam problem: Given distributions of force per unit length $q_n = q_n(s)$, couple per unit length $l_b = l_b(s)$, $s \in (0, L)$, and natural or essential boundary conditions at each end of the beam, find the displacement field $u_n = u_n(s)$ and the rotation $\varphi = \varphi(s)$ such that (11) and (12) are satisfied for all $s \in (0, L)$.
A model expressing rigidity in shearing is obtained by formally letting the shear stiffness $k$ approach infinity. This implies that $\varepsilon_n \to 0$ when $k \to \infty$, and we obtain
\[
\varphi = \frac{\partial u_n}{\partial s}. \tag{13}
\]
This condition is now used to eliminate $\varphi$ from all relevant equations and we obtain the Euler-Bernoulli differential equation
\[
-\frac{\partial^2}{\partial s^2} \left( (EI) \frac{\partial^2 u_n}{\partial s^2} \right) + q_n - \frac{\partial l_b}{\partial s} = 0. \tag{14}
\]

**Euler-Bernoulli beam problem:** Given distributions of force per unit length $q_n = q_n(s)$ and couple per unit length $l_b = l_b(s)$, $s \in (0, L)$, and at each end of the beam natural or essential boundary conditions, find the displacement field $u_n = u_n(s)$ such that (14) is satisfied for all $s \in (0, L)$. 

A reversed type of beam behavior is when the bending stiffness \((EI)\) goes to infinity while the shear stiffness \(k\) remains finite. This reduces (11) to

\[
\frac{\partial}{\partial s} \left( k \frac{\partial u_n}{\partial s} \right) + q_n = 0. \tag{15}
\]

**Shear beam problem:** Given a distribution of force per unit length \(q_n = q_n(s), s \in (0, L)\), and at each end of the beam a natural or essential boundary condition, find the displacement field \(u_n = u_n(s)\) such that (15) is satisfied for all \(s \in (0, L)\).
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Equilibrium equation

\[ \text{div } \mathbf{T} + \mathbf{b} = \mathbf{0} \quad \text{in} \quad B, \]  
\[ \mathbf{s} = \mathbf{T} \mathbf{n} \quad \text{on} \quad \partial B, \]  
(16)  
(17)

Geometric equation

\[ E = \frac{1}{2} \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right), \]  
(18)

These two equations are such that \( W^e = W^i \) where

\[ W^e = \int_B \mathbf{b} \cdot \mathbf{u} \, dV + \int_{\partial B} \mathbf{s} \cdot \mathbf{u} \, dA. \]

\[ W^i = \int_B \mathbf{T} : \mathbf{E} \, dV, \quad \mathbf{T} : \mathbf{E} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} \varepsilon_{ij}. \]
The constitutive law is any linear relation between $T$ and $E$. In general such a linear relation is described by $6 \times 6 = 36$ constants. However, if the material is isotropic we only need 2 constants!! The linear relation is then written as:

$$ T = \lambda (\text{tr } E) I + 2\mu E, $$

where $\lambda$ and $\mu$ are Lamé’s elasticity coefficients.

That a material is isotropic means that it has the same response in all directions. This obviously does not hold for, say, wood, or any other material with fiber structure.